

# LINEAR INDEPENDENCE OF MONOMIALS OF MULTIZETA VALUES IN POSITIVE CHARACTERISTIC

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**ABSTRACT.** In this paper, we study transcendence theory for Thakur multizeta values in positive characteristic. We prove an analogue of the strong form of Goncharov's conjecture. We also establish the same result for Carlitz multiple polylogarithms at nontrivial algebraic points.

## 1. INTRODUCTION

**1.1. Classical multiple zeta values.** Multiple zeta values (abbreviated as MZVs) are real numbers defined by Euler:

$$\zeta(s_1, \dots, s_r) := \sum_{n_1 > \dots > n_r \geq 1} \frac{1}{n_1^{s_1} \dots n_r^{s_r}},$$

where  $s_1, \dots, s_r$  are positive integers with  $s_1 \geq 2$ . Here  $r$  is called the depth and  $\sum_{i=1}^r s_i$  is called the weight of the MZV  $\zeta(s_1, \dots, s_r)$ . These values are generalizations of the Riemann zeta function at positive integers, and have been much studied in recent years because of various points of view of their interesting properties. For example, they occur as periods of the mixed Tate motives, and they occur as values of Feynman integrals in quantum field theory. We refer the reader to the papers on this subject by Brown, Deligne, Drinfeld, Goncharov, Hoffman, Terasoma, Waldschmidt, Zagier etc. See also the recent advances by Brown [B12] and Zagier [Z12].

It is natural to ask the transcendence nature of these MZVs. However, it is still an open problem although one knows the transcendence of the Riemann zeta function at even positive integers because of Euler's formula. The main motivation of the study in this paper is from the important conjecture given by Goncharov [Gon97]: let  $\mathfrak{Z}$  be the  $\mathbb{Q}$ -algebra generated by MZVs and let  $\mathfrak{Z}_w$  be the  $\mathbb{Q}$ -vector space spanned by the weight  $w$  MZVs, then

$$\mathfrak{Z} = \mathbb{Q} \oplus_{w \geq 2} \mathfrak{Z}_w.$$

That is, conjecturally the  $\mathbb{Q}$ -algebra  $\mathfrak{Z}$  is a graded algebra (graded by weights). The following conjecture (folklore) is a stronger form of Goncharov's conjecture.

**Conjecture 1.1.1.** *Let  $\overline{\mathfrak{Z}}$  be the  $\overline{\mathbb{Q}}$ -algebra generated by MZVs, and let  $\overline{\mathfrak{Z}}_w$  be the  $\overline{\mathbb{Q}}$ -vector space spanned by the weight  $w$  MZVs for  $w \geq 2$ . Then one has that  $\overline{\mathfrak{Z}} = \overline{\mathbb{Q}} \oplus_{w \geq 2} \overline{\mathfrak{Z}}_w$  and  $\overline{\mathfrak{Z}}$  is defined over  $\mathbb{Q}$ .*

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*Date:* August 13, 2012.

*2000 Mathematics Subject Classification.* Primary 11J91, 11J93.

*Key words and phrases.* Multizeta values; Transcendence; Linear independence; Carlitz multiple polylogarithms.

The author was partially supported by NSC Grant 100-2115-M-007-010-MY3.

The author thanks HUST and IHES, where he worked on this paper.

In other words, to prove their linear independence over  $\mathbb{Q}$ , one could adopt a strategy of proving linear independence over  $\overline{\mathbb{Q}}$  for these special values although it is still wild open. The primary purpose of this article is to prove an analogue of the conjecture above in the setting of multizeta values in positive characteristic.

**1.2. Thakur multizeta values.** In analogy with the classical MZVs, in his seminal work [To4] Thakur studied the characteristic  $p$  multizeta values (abbreviated as MZVs) in  $k_\infty^\times$ , where  $k$  is the rational function field  $\mathbb{F}_q(\theta)$  over a finite field  $\mathbb{F}_q$  and  $k_\infty$  is the completion of  $k$  at  $\infty$ , which is the zero divisor of  $1/\theta$ . (See § 2.2 for the definition of MZVs). These MZVs are generalizations of the Carlitz zeta values at positive integers [Ca35], and they occur as periods of mixed Carlitz-Tate motives (explicitly constructed) by the work of Anderson-Thakur [AT09]. Notice that the weight one MZV is just the Carlitz zeta value at 1, which exists in this non-archimedean field setting. Note further that Thakur [T10] showed that a product of two multizeta values of weight  $w_1$  and  $w_2$  can be expressed as an  $\mathbb{F}_p$ -linear combination of MZVs of weight  $w_1 + w_2$  (see [LR11] for the explicit expressions), where  $p$  is the characteristic of  $\mathbb{F}_q$ .

The first main result in this paper is to prove a precise function field analogue of Conjecture 1.1.1 (stated as Theorem 2.2.1). That is, the  $\bar{k}$ -algebra generated by all MZVs forms a graded algebra (graded by weights) defined over  $k$ . As consequences, one has:

- Each nontrivial monomial of MZVs is transcendental over  $k$ ;
- The ratio of two different weight nontrivial monomials of MZVs is transcendental over  $k$ .

The results above generalize the work of Yu [Yu91, Yu97] for the depth one case, and the work of Thakur [To9b] on the transcendence of some specific MZVs. We further derive the following consequences stated as Theorem 2.3.2 and Corollary 2.3.3:

- Let  $Z_1$  and  $Z_2$  be two MZVs of the same weight. Then either  $Z_1/Z_2 \in k$  or  $Z_1$  and  $Z_2$  are algebraically independent over  $k$ .
- Let  $Z$  be a MZV of weight  $w$ . Then either  $Z/\tilde{\pi}^w$  is in  $k$  or  $Z$  is algebraically independent from  $\tilde{\pi}$ .

Here  $\tilde{\pi}$  is a fundamental period of the Carlitz module, which plays the analogous role of  $2\pi\sqrt{-1}$  for the multiplicative group  $\mathbb{G}_m$ . The last property listed above is called *Euler dichotomy* phenomena (see § 2.3). In particular, every multizeta value of “odd” weight  $w$  (i.e.,  $(q-1) \nmid w$ ) is algebraically independent from  $\tilde{\pi}$ .

The main goal of transcendence theory for MZVs is to determine all the  $\bar{k}$ -algebraic relations among the MZVs. However, in contrast to the classical case, a nice description of the full set of identities satisfied by MZVs is not known yet (see [AT09, T10]). As all  $\bar{k}$ -algebraic relations among the MZVs are  $\bar{k}$ -linear relations among the monomials of MZVs, Theorem 2.2.1 has shown that all  $\bar{k}$ -algebraic relations are coming from the  $k$ -linear relations among the same weight monomials of MZVs. However, there still remains the key problem of finding all the  $k$ -linear relations among the same weight monomials of MZVs. Note that unlike the classical case, the prime field is not  $k$ , analog of  $\mathbb{Q}$ , but  $\mathbb{F}_p$ , and  $\mathbb{F}_p$ -relations are understood in [T10].

**1.3. Multiple polylogarithms.** Classical multiple polylogarithms with several variables are generalizations of polylogarithms and their specializations at  $(1, \dots, 1)$  give the MZVs. This phenomena becomes nontrivially delicate in the function field setting. In [AT90], Anderson-Thakur established that Carlitz zeta value at  $n \in \mathbb{N}$  (ie., the multizeta

value of weight  $n$  and depth one) can be expressed as a  $k$ -linear combination of the  $n$ -th Carlitz polylogarithm at integral points.

In this article, we define the Carlitz multiple polylogarithms (abbreviated as CMPLs), then the work of Anderson-Thakur is extended to multizeta values. From the definition, one sees that these CMPLs satisfy the stuffle relations. The second main result in this article is to prove an analogue of Conjecture 1.1.1 for the CMPLs at algebraic points. The result is stated as Theorem 6.4.3, which implies that the  $\bar{k}$ -algebra generated by CMPLs at nontrivial algebraic points forms a graded algebra defined over  $k$ . As consequences, one further has:

- Each nontrivial monomial of CMPLs at nontrivial algebraic points is transcendental over  $k$ ;
- The ratio of two different weight nontrivial monomials of CMPLs at nontrivial algebraic points is transcendental over  $k$ .
- Let  $Z_1$  and  $Z_2$  be two nonzero values which are CMPLs at algebraic points. If  $Z_1$  and  $Z_2$  are of the same weight, then either  $Z_1/Z_2 \in k$  or  $Z_1$  and  $Z_2$  are algebraically independent over  $k$ .

Let  $\overline{\mathfrak{M}}$  be the  $\bar{k}$ -algebra generated by all CMPLs at algebraic points. As Theorem 6.4.3 implies that all the  $\bar{k}$ -polynomial relations among the CMPLs at algebraic points are homogenous over  $k$ , it is natural to ask how to describe the  $k$ -linear relations among the same weight monomials and we wish to tackle this problem in the future. Figuring out the problem above would be helpful to understand the relations among MZVs as we have  $\overline{\mathfrak{Z}} \subseteq \overline{\mathfrak{M}}$ .

**1.4. Outline and some remarks.** In § 2, we fix our notation and state our result on multizeta values. Based on the work of [AT09], one is able to create Frobenius difference equations for which the specialization of the solution functions gives the given MZVs. We observe that the case of CMPLs at nontrivial algebraic points shares the same property as above. Hence we shall say that such values have the *MZ* property (see Definition 3.4.1).

In § 3, we state a general linear independence result for the nonzero values having the *MZ* property, which is stated as Theorem 3.4.4. We give a proof of Theorem 3.4.4 in § 4, and then show in § 5 and § 6 that the multizeta values and the CMPLs at nontrivial algebraic points have the *MZ* property, and hence appeal to Theorem 3.4.4 showing the desired results.

We mention that the tools of proving algebraic independence using  $t$ -motives introduced by Anderson [A86] come from Papanikolas [Po8], which can be regarded as a function field analogue of Grothendieck's periods conjecture. Using these tools, one has the algebraic independence results on Carlitz zeta values [CY07], Drinfeld logarithms at algebraic points [CP12] etc. Although one is able to construct suitable  $t$ -motives so that the given multizeta values or CMPLs at nontrivial algebraic points occur as periods of the  $t$ -motives (cf. [AT09] and § 6.4), to obtain the more comprehensive algebraic independence results on multizeta values or CMPLs at algebraic points via Papanikolas' theory one has to compute the dimension of the relevant  $t$ -motivic Galois group. However, the dimension of such  $t$ -motivic Galois group relies on the information of the periods of the  $t$ -motive, which is closely related to the description of the rich identities that multizeta values or CMPLs at algebraic points satisfy. Hence it would be difficult to compute the dimension of the Galois group in question at this moment.

The overall strategy of showing Theorem 3.4.4 is to use the criterion established by Anderson-Brownawell-Papanikolas [ABPo4] (abbreviated as ABP-criterion). We apply the ABP-criterion to lift the given  $\bar{k}$ -linear relations among the special values in question to the  $\bar{k}[t]$ -linear relations among the solution functions. Then we analyze the coefficients (functions) as well as the solution functions to show the desired result. Finally, we emphasize that in this setting using the ABP-criterion opens a door towards the general linear independence results in question, and it enables one to avoid some difficulties occurring in the computation of the relevant Galois groups via Papanikolas' theory.

## 2. MAIN RESULT FOR MULTIZETA VALUES

**2.1. Notation.** In this paper, we adopt the following notation.

$\mathbb{F}_q$	= the finite field with $q$ elements, for $q$ a power of a prime number $p$ .
$\theta, t$	= independent variables.
$A$	= $\mathbb{F}_q[\theta]$ , the polynomial ring in the variable $\theta$ over $\mathbb{F}_q$ .
$A_+$	= set of monic polynomials in $A$ .
$k$	= $\mathbb{F}_q(\theta)$ , the fraction field of $A$ .
$k_\infty$	= $\mathbb{F}_q((1/\theta))$ , the completion of $k$ with respect to the place at infinity.
$\overline{k_\infty}$	= a fixed algebraic closure of $k_\infty$ .
$\bar{k}$	= the algebraic closure of $k$ in $\overline{k_\infty}$ .
$\mathbb{C}_\infty$	= the completion of $\overline{k_\infty}$ with respect to the canonical extension of $\infty$ .
$ \cdot _\infty$	= a fixed absolute value for the completed field $\mathbb{C}_\infty$ so that $ \theta _\infty = q$ .
$\deg$	= function assigning to $x \in k_\infty$ its degree in $\theta$ .
$\mathbb{C}_\infty[[t]]$	= ring of formal power series in $t$ over $\mathbb{C}_\infty$ .

**2.2. Multizeta values.** Given any  $s \in \mathbb{N}$  and nonnegative integer  $d$ , we define the power sum:

$$S_d(s) := \sum_{\substack{a \in A_+ \\ \deg a = d}} \frac{1}{a^s}.$$

In analogy with the classical multiple zeta values, Thakur [To4] studied the following multizeta values (which we abbreviate as MZVs): for any  $r$ -tuple  $(s_1, \dots, s_r) \in \mathbb{N}^r$ ,

$$\zeta(s_1, \dots, s_r) := \sum_{d_1 > \dots > d_r \geq 0} S_{d_1}(s_1) \cdots S_{d_r}(s_r) = \sum \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in k_\infty,$$

where the second sum is over  $(a_1, \dots, a_r) \in A_+^r$  with  $\deg a_i$  strictly decreasing. We call this MZV having *depth*  $r$  and *weight*  $\sum_{i=1}^r s_i$ . In the case of  $r = 1$ , the values above are the Carlitz zeta values at positive integers. Note that each MZV is nonzero by the work of Thakur [To9a]. Note further that there is no natural order on polynomials in contrast to integers, so unlike the classical case, it is not immediately clear that the span of MZVs is an algebra, but this together with period interpretation was conjectured and then proved in [To9b, AT09, T10].

Let  $Z_1, \dots, Z_n$  be MZVs of weights  $w_1, \dots, w_n$  respectively. For nonnegative integers  $m_1, \dots, m_n$ , not all zero, we define the (total) weight of the monomial  $Z_1^{m_1} \cdots Z_n^{m_n}$  to be

$$\sum_{i=1}^n m_i w_i.$$

Let  $\overline{\mathfrak{Z}}_w$  be the  $\bar{k}$ -vector space spanned by weight  $w$  MZVs, and let  $\overline{\mathfrak{Z}}$  be the  $\bar{k}$ -algebra generated by all MZVs. Our first main result is stated as follows, and its proof is given in § 5.4.

**Theorem 2.2.1.** *Let  $w_1, \dots, w_\ell$  be  $\ell$  distinct positive integers. Let  $V_i$  be a finite set consisting of some monomials of multizeta values of total weight  $w_i$  for  $i = 1, \dots, \ell$ . If  $V_i$  is a linearly independent set over  $k$ , then the set*

$$\{1\} \bigcup_{i=1}^{\ell} V_i$$

*is linearly independent over  $\bar{k}$ . In particular, we have*

$$\overline{\mathfrak{Z}} = \bar{k} \oplus_{w \in \mathbb{N}} \overline{\mathfrak{Z}}_w \text{ and } \overline{\mathfrak{Z}} \text{ is defined over } k.$$

Note that by [To9b] we have  $\overline{\mathfrak{Z}}_w \overline{\mathfrak{Z}}_{w'} \subseteq \overline{\mathfrak{Z}}_{w+w'}$  and so the theorem above implies that  $\overline{\mathfrak{Z}}$  forms a graded algebra (graded by weights) defined over  $k$ .

**Corollary 2.2.2.** *Each nontrivial monomial of multizeta values is transcendental over  $k$ .*

**Corollary 2.2.3.** *The ratio of two different weight nontrivial monomials of multizeta values is transcendental over  $k$ .*

**2.3. Euler dichotomy.** Let  $\tilde{\pi}$  be a fundamental period of the Carlitz module defined in (3.1.3). In analogy with Euler's formula for the classical Riemann zeta function at even positive integers, Carlitz [Ca35] showed that for a positive integer  $n$  divisible by  $q - 1$  one has

$$(2.3.1) \quad \zeta(n) = c_n \tilde{\pi}^n,$$

where  $c_n$  is in  $k^\times$  and can be expressed in terms of Bernoulli-Carlitz numbers and Carlitz factorials (cf. [Goss96, To4]). We shall call a positive integer  $n$  "even" if  $(q - 1) | n$ ; otherwise it is called "odd". Therefore, we shall call a weight  $w$  multizeta value  $Z$  *Eulerian* if the ratio  $Z / \tilde{\pi}^w$  is in  $k$ . Using Theorem 2.2.1 we have following result.

**Theorem 2.3.2.** *Let  $Z_1, Z_2$  be two multizeta values of the same weight  $w$ . Then either the ratio  $Z_1 / Z_2$  is in  $k$  or  $Z_1$  and  $Z_2$  are algebraically independent over  $k$ .*

*Proof.* Suppose that  $Z_1 / Z_2 \notin k$ . Thus, by Theorem 2.2.1 the ratio  $Z_1 / Z_2$  is transcendental over  $k$ . If  $Z_1$  and  $Z_2$  are algebraically dependent over  $k$ , then by Theorem 2.2.1 there exists a homogenous polynomial  $F(X, Y) \in k[X, Y]$  of positive degree so that  $F(Z_1, Z_2) = 0$ . Let  $d$  be the total degree of  $F$ . Then dividing the equation  $F(Z_1, Z_2) = 0$  by  $Z_2^d$  we see that the ratio  $Z_1 / Z_2$  satisfies a nontrivial polynomial over  $k$ , whence a contradiction.  $\square$

Let  $Z$  be a MZV of weight  $w$ . If the ratio  $Z / \tilde{\pi}^w$  is algebraic over  $k$ , then in Corollary 5.3.3 we show the descent property of  $Z / \tilde{\pi}^w$ , and hence we derive the following *Euler dichotomy* phenomenon from Theorem 2.3.2.

**Corollary 2.3.3.** *Every multizeta value is either Eulerian or is algebraically independent from  $\tilde{\pi}$ . In particular, every multizeta value of "odd" weight  $w$  is algebraically independent from  $\tilde{\pi}$ .*

*Proof.* Let  $Z$  be a multizeta value of weight  $w$  and suppose that  $Z / \tilde{\pi}^w \notin k$ . Thus by Corollary 5.3.3 we have  $Z / \tilde{\pi}^w \notin \bar{k}$ . It follows from (2.3.1) that  $Z^{q-1} / \zeta(w(q-1)) \notin \bar{k}$ . So Theorem 2.3.2 implies the algebraic independence of  $Z^{q-1}$  and  $\zeta(w(q-1))$  over  $k$ , whence the algebraic independence of  $Z$  and  $\tilde{\pi}^w$  (because of (2.3.1)), which implies the algebraic independence of  $Z$  and  $\tilde{\pi}$ .

To show the second assertion, we need only consider  $q > 2$  since all positive integers are “even” in the case of  $q = 2$ . Note that for  $q > 2$ , one observes that from the definition (3.1.3) we have  $\tilde{\pi}^w \notin k_\infty$  if  $w$  is not a multiple of  $(q - 1)$ . Since every MZV is in  $k_\infty$ , every MZV of “odd” weight is not Eulerian and so the assertion follows from the previous one.  $\square$

*Remark 2.3.4.* Thakur [To4, Thm. 5.10.12] first observed that  $\zeta(2, 1)$  and  $\zeta(1, 2)$  are not Eulerian in the case of  $q = 2$  (note that MZVs and  $\tilde{\pi}$  are belong to  $k_\infty$  in this case), and hence they are algebraically independent from Carlitz zeta values when  $p = 2$ . In other words, there is an MZV which is algebraically independent from all Carlitz zeta values. This gives a positive answer of the analogous question in [Andréo4, p. 231].

### 3. LINEAR INDEPENDENCE OF SPECIAL VALUES OCCURRING FROM DIFFERENCE EQUATIONS

In this section, the main goal is to establish a linear independence result of certain special values occurring from difference equations which is applied to prove Theorem 2.2.1.

**3.1. Twisting operators.** For any integer  $n$ , we define the  $n$ -fold twisting on the field of Laurent series  $\mathbb{C}_\infty((t))$ :

$$\begin{aligned} \mathbb{C}_\infty((t)) &\rightarrow \mathbb{C}_\infty((t)) \\ f := \sum a_i t^i &\mapsto f^{(n)} := \sum a_i^{q^n} t^i. \end{aligned}$$

We note that

$$(3.1.1) \quad \left\{ f \in \bar{k}(t); f^{(-1)} = f \right\} = \mathbb{F}_q(t).$$

Note further that the  $n$ -fold twisting is extended to act on  $\text{Mat}_{m \times n}(\mathbb{C}_\infty((t)))$  entrywise.

Throughout this paper, we fix a  $(q - 1)$ -th root of  $-\theta$  and denote it by  $\tilde{\theta}$ . The function

$$\Omega(t) := \tilde{\theta}^{-q} \prod_{i=1}^{\infty} \left( 1 - \frac{t}{\theta^{q^i}} \right)$$

has a power series expansion in  $t$ , and is entire on  $\mathbb{C}_\infty$  and satisfies the following difference equation:

$$(3.1.2) \quad \Omega^{(-1)}(t) = (t - \theta)\Omega(t).$$

Moreover, the following value

$$(3.1.3) \quad \tilde{\pi} := \frac{1}{\Omega(\theta)}$$

is a fundamental period of the Carlitz module (cf. [AT90, ABP04]).

**3.2. ABP-criterion.** We define  $\mathcal{E}$  to be the ring consisting of formal power series

$$\sum_{n=0}^{\infty} a_n t^n \in \bar{k}[[t]]$$

such that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|_\infty} = 0, \quad [k_\infty(a_0, a_1, a_2, \dots) : k_\infty] < \infty.$$

Then any  $f$  in  $\mathcal{E}$  has an infinite radius of convergence with respect to  $|\cdot|_\infty$  and has the property that  $f(\alpha) \in \overline{k_\infty}$  for any  $\alpha \in \overline{k_\infty}$ .

To state and show the main result of this section, we shall review the ABP-criterion.

**Theorem 3.2.1.** (Anderson-Brownawell-Papanikolas, [ABPo4, Thm. 3.1.1]) *Fix a matrix  $\Phi \in \text{Mat}_\ell(\bar{k}[t])$  so that  $\det \Phi = c(t - \theta)^s$  for some  $c \in \bar{k}^\times$  and some nonnegative integer  $s$ . Suppose that there exists a vector  $\psi \in \text{Mat}_{\ell \times 1}(\mathcal{E})$  satisfying that*

$$\psi^{(-1)} = \Phi \psi.$$

*Then for each row vector  $\rho \in \text{Mat}_{1 \times \ell}(\bar{k})$  such that  $\rho \psi(\theta) = 0$ , there exists a vector  $P \in \text{Mat}_{1 \times \ell}(\bar{k}[t])$  such that*

$$P(\theta) = \rho \text{ and } P\psi = 0.$$

The spirit of the ABP-criterion is that every  $\bar{k}$ -linear relation among the entries of  $\psi(\theta)$  can be lifted to a  $\bar{k}[t]$ -linear relation among the entries of  $\psi$ .

*Remark 3.2.2.* In [Co9], a refined version of the ABP-criterion which relaxes the condition of  $\Phi$  and the specialization of  $\psi$  at more algebraic points is given. But here the ABP-criterion is sufficient for our proof.

**3.3. Some notation.** Considering square matrices  $M_i \in \text{Mat}_{n_i}(\mathbb{C}_\infty[[t]])$  with  $n_i \geq 2$  for  $i = 1, \dots, \ell$ , we define  $\oplus_{i=1}^\ell M_i$  to be the block diagonal matrix

$$\begin{pmatrix} M_1 & & \\ & \ddots & \\ & & M_\ell \end{pmatrix}.$$

For column vectors  $\mathbf{v}_1, \dots, \mathbf{v}_m$  with entries in  $\mathbb{C}_\infty[[t]]$ , we define  $\oplus_{i=1}^m \mathbf{v}_i$  to be the column vector

$$(\mathbf{v}_1^{\text{tr}}, \dots, \mathbf{v}_m^{\text{tr}})^{\text{tr}}.$$

### 3.4. A linear independence result.

**Definition 3.4.1.** A nonzero element  $Z \in k^\times$  is said to have the MZ (Multizeta) property with weight  $w$  if there exists  $\Phi \in \text{Mat}_d(\bar{k}[t])$  and  $\psi \in \text{Mat}_{d \times 1}(\mathcal{E})$  with  $d \geq 2$  so that

- (1)  $\psi^{(-1)} = \Phi \psi$  and  $\Phi$  satisfies the conditions of the ABP-criterion;
- (2) The last column of  $\Phi$  is of the form  $(0, \dots, 1)^{\text{tr}}$  (whose entries are zero except the last entry being 1);
- (3)  $\psi(\theta)$  is of the form (with specific first and last entries):

$$\psi(\theta) = \begin{pmatrix} 1/\tilde{\pi}^w \\ \vdots \\ cZ/\tilde{\pi}^w \end{pmatrix}$$

for some  $c \in k^\times$ ;

- (4) for any positive integer  $N$ ,  $\psi(\theta^{q^N})$  is of the form:

$$\psi(\theta^{q^N}) = \begin{pmatrix} 0 \\ \vdots \\ (cZ/\tilde{\pi}^w)^{q^N} \end{pmatrix}$$

(whose entries are zero except the last entry).

*Remark 3.4.2.* We will see from Theorem 3.4.4 that any nonzero  $Z$  having the MZ property has a unique weight.

**Proposition 3.4.3.** *Let  $Z_1, \dots, Z_n$  be nonzero values having the MZ property with weights  $w_1, \dots, w_n$  respectively. For nonnegative integers  $m_1, \dots, m_n$ , not all zero, the monomial*

$$Z_1^{m_1} \cdots Z_n^{m_n}$$

*has the MZ property with weight  $\sum_{i=1}^n m_i w_i$ .*

*Proof.* We consider the Kronecker product:

$$\Phi := \Phi_1^{\otimes m_1} \otimes \cdots \otimes \Phi_n^{\otimes m_n} \text{ and } \psi := \psi_1^{\otimes m_1} \otimes \cdots \otimes \psi_n^{\otimes m_n}.$$

Then one has  $\psi^{(-1)} = \Phi\psi$ . Since each triple  $(\Phi_i, \psi_i, Z_i)$  satisfies (1) – (4) of Definition 3.4.1, one sees that the triple  $(\Phi, \psi, Z_1^{m_1} \cdots Z_n^{m_n})$  satisfies the conditions of Definition 3.4.1 and hence  $Z_1^{m_1} \cdots Z_n^{m_n}$  has the MZ property with weight  $\sum_{i=1}^n m_i w_i$ .  $\square$

The main result in this section is stated as follows, and its proof occupies the next section.

**Theorem 3.4.4.** *Let  $w_1, \dots, w_\ell$  be  $\ell$  distinct positive integers. Let  $V_i$  be a finite set of some nonzero values having the MZ-property with weight  $w_i$ , and suppose that  $V_i$  is a linearly independent set over  $k$  for  $i = 1, \dots, n$ . Then the union*

$$\{1\} \bigcup_{i=1}^{\ell} V_i$$

*is a linearly independent set over  $\bar{k}$ .*

#### 4. PROOF OF THEOREM 3.4.4

In this section, we give a proof of Theorem 3.4.4. Let notation and assumptions be given in Theorem 3.4.4. Without loss of generality, we may assume that  $w_1 > \cdots > w_\ell$ . Suppose on the contrary that the set

$$\{1\} \bigcup_{i=1}^{\ell} V_i$$

is linearly dependent over  $\bar{k}$ . By induction on the weight, we may further assume that there are nontrivial  $\bar{k}$ -linear relations connecting  $V_1$  and  $\{1\} \bigcup_{i=2}^{\ell} V_i$ . Under such hypotheses, we complete the proof in the following two steps.

**Step I :** We show that  $V_1$  is a linearly dependent set over  $\bar{k}$ ;

**Step II :** We show that  $V_1$  is a linearly dependent set over  $k$ , whence a contradiction.

**4.1. Proof of Step I.** In this step, our goal is to show that  $V_1$  is a linearly dependent set over  $\bar{k}$ . Let  $V_i$  consist of  $\{Z_{i1}, \dots, Z_{im_i}\}$  of the same weight  $w_i$  for  $i = 1, \dots, \ell$ . For  $1 \leq i \leq \ell$ , since  $Z_{ij}$  has the MZ property there exists  $\Phi_{ij} \in \text{Mat}_{d_{ij}}(\bar{k}[t])$  and  $\psi_{ij} \in \text{Mat}_{d_{ij} \times 1}(\mathcal{E})$  (with  $d_{ij} \geq 2$ ) satisfying Definition 3.4.1 (corresponding to the  $Z_{ij}$ ) for  $j = 1, \dots, m_i$ .

Define the block diagonal matrix

$$\tilde{\Phi} := \oplus_{i=1}^{\ell} \left( \oplus_{j=1}^{m_i} (t - \theta)^{w_1 - w_i} \Phi_{ij} \right)$$

and the column vector

$$\tilde{\psi} := \oplus_{i=1}^{\ell} \left( \oplus_{j=1}^{m_i} \Omega^{w_1 - w_i} \psi_{ij} \right).$$



Then one has  $\tilde{\psi}^{(-1)} = \tilde{\Phi}\tilde{\psi}$ . From Definition 3.4.1, it follows that  $\tilde{\psi}(\theta)$  is of the form:

$$\tilde{\psi}(\theta) = \oplus_{j=1}^{m_1} \begin{pmatrix} 1/\tilde{\pi}^{w_1} \\ \vdots \\ (c_{1j}Z_{1j}/\tilde{\pi}^{w_1}) \end{pmatrix} \oplus_{i=2}^{\ell} \left( \oplus_{j=1}^{m_i} \begin{pmatrix} 1/\tilde{\pi}^{w_1} \\ \vdots \\ c_{ij}Z_{ij}/\tilde{\pi}^{w_1} \end{pmatrix} \right).$$

Note that since  $w_1 > w_i$  for  $2 \leq i \leq \ell$  and  $\Omega(t)$  has simple zero at  $t = \theta^{q^N}$  for  $N \in \mathbb{N}$ , for any positive integer  $N$  we see that  $\tilde{\psi}(\theta^{q^N})$  is of the form

$$(4.1.1) \quad \tilde{\psi}(\theta^{q^N}) = \oplus_{j=1}^{m_1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (c_{1j}Z_{1j}/\tilde{\pi}^{w_1})^{q^N} \end{pmatrix} \oplus_{i=2}^{\ell} \left( \oplus_{j=1}^{m_i} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \right).$$

Since by assumption  $\{1\} \cup_{i=1}^{\ell} \{Z_{i1}, \dots, Z_{im_i}\}$  is linearly dependent over  $\bar{k}$ , there exists a nonzero vector  $\rho$  for which  $\rho\tilde{\psi}(\theta) = 0$ . We write

$$\rho = (\mathbf{v}_{11}, \dots, \mathbf{v}_{1m_1}, \dots, \mathbf{v}_{\ell 1}, \dots, \mathbf{v}_{\ell m_{\ell}}),$$

where  $\mathbf{v}_{ij} \in \text{Mat}_{1 \times d_{ij}}(\bar{k})$  for  $1 \leq j \leq m_i$ ,  $1 \leq i \leq \ell$ . Since we assume that there are nontrivial  $\bar{k}$ -linear relations connecting  $V_1$  and  $\{1\} \cup_{i=2}^{\ell} V_i$ , the last entry of  $\mathbf{v}_{1s}$  is nonzero for some  $1 \leq s \leq m_1$ .

By Theorem 3.2.1 for each  $1 \leq i \leq \ell$  there exists  $\mathbf{f}_{ij} \in \text{Mat}_{1 \times d_{ij}}(\bar{k}[t])$  (for  $j = 1, \dots, m_i$ ) so that  $\mathbf{F} := (\mathbf{f}_{11}, \dots, \mathbf{f}_{1m_1}, \dots, \mathbf{f}_{\ell 1}, \dots, \mathbf{f}_{\ell m_{\ell}})$  satisfies

$$\mathbf{F}\tilde{\psi} = 0 \text{ and } \mathbf{F}(\theta) = \rho.$$

Since by hypothesis the last entry of  $\mathbf{v}_{1s}$  is nonzero, the last entry of  $\mathbf{f}_{1s}$  is a nontrivial polynomial. We pick an integer  $N$  sufficiently large for which the last entry of  $\mathbf{f}_{1s}$  is non-vanishing at  $t = \theta^{q^N}$ . Specializing the equation  $\mathbf{F}\tilde{\psi} = 0$  at  $t = \theta^{q^N}$  and using (4.1.1) gives rise to a nontrivial  $\bar{k}$ -linear relation among

$$Z_{11}^{q^N}, \dots, Z_{1m_1}^{q^N}.$$

Since our field is of characteristic  $p$ , by taking the  $q^N$ -th root from the  $\bar{k}$ -linear relation above we obtain a nontrivial  $\bar{k}$ -linear relation among the weight  $w_1$  values  $\{Z_{11}, \dots, Z_{1m_1}\}$ , as claimed.

**4.2. Proof of Step II.** In this step, our goal is to show that  $V_1$  is a linearly dependent set over  $k$ , whence a contradiction and thus we complete the proof of Theorem 3.4.4. According to Step I above, we have shown that  $V_1$  is linearly dependent over  $\bar{k}$ . Without confusion with the notation of double index in Step I, for simplicity we write  $V_1 = \{Z_1, \dots, Z_m\}$ , and without loss of generality we may assume that  $m \geq 2$  and

$$\dim_{\bar{k}} \bar{k}\text{-Span} \{V_1\} = m - 1.$$

Again for simplicity and without confusion with the double index above, we let  $\Phi_j \in \text{Mat}_{d_j}(\bar{k}[t])$  and  $\psi_j \in \text{Mat}_{d_j \times 1}(\mathcal{E})$  (with  $d_j \geq 2$ ) be associated to the value  $Z_j$  having the MZ property with weight  $w_1$  for  $j = 1, \dots, m$ .

Define the block diagonal matrix

$$\Phi := \oplus_{j=1}^m \Phi_j$$

and define the column vector

$$\psi := \oplus_{j=1}^m \psi_j.$$

Notice that

$$(4.2.1) \quad \psi(\theta) = \oplus_{j=1}^m \begin{pmatrix} 1/\tilde{\pi}^{w_1} \\ \vdots \\ c_j Z_j / \tilde{\pi}^{w_1} \end{pmatrix}$$

for some  $c_j \in k^\times$ , and for  $N \in \mathbb{N}$  we have

$$(4.2.2) \quad \psi(\theta^{q^N}) = \oplus_{j=1}^m \begin{pmatrix} 0 \\ \vdots \\ (c_j Z_j / \tilde{\pi}^{w_1})^{q^N} \end{pmatrix}.$$

Without loss of generality, we may assume that  $Z_1 \in \bar{k}\text{-Span}\{Z_2, \dots, Z_m\}$ , and so by hypothesis  $\{Z_2, \dots, Z_m\}$  is linearly independent over  $\bar{k}$ . By the ABP-criterion (cf. Theroem 3.2.1) there exists vectors  $\mathbf{f}_j = (\mathbf{f}_{j1}, \dots, \mathbf{f}_{jd_j}) \in \text{Mat}_{1 \times d_j}(\bar{k}[t])$  for  $j = 1, \dots, m$  so that if we put  $\mathbf{F} := (\mathbf{f}_1, \dots, \mathbf{f}_m)$  then we have

$$(4.2.3) \quad \mathbf{F}\psi = 0, \mathbf{f}_{1d_1}(\theta) = 1 \text{ and } \mathbf{f}_{jh}(\theta) = 0 \text{ for all } 1 \leq h < d_j.$$

We divide the vector  $\mathbf{F}$  by  $\mathbf{f}_{1d_1}$ , and write  $\mathbf{G} := \frac{1}{\mathbf{f}_{1d_1}} \mathbf{F}$ . Let  $d := \sum_{j=1}^m d_j$ . Note that the vector  $\mathbf{G}$  is of the form

$$\mathbf{G} = (\mathbf{g}_{11}, \dots, 1, \dots, \mathbf{g}_{m1}, \dots, \mathbf{g}_{md_m}) \in \text{Mat}_{1 \times d}(\bar{k}(t)),$$

where 1 is corresponding to the  $(1, d_1)$ -entry of  $\mathbf{G}$ , and we have

$$(4.2.4) \quad \mathbf{G}\psi = 0 \text{ and } \mathbf{g}_{jh}(\theta) = 0 \text{ for all } 1 \leq h < d_j.$$

We use the  $(-1)$ -fold twisting action on  $\mathbf{G}\psi = 0$ , and so obtain  $\mathbf{G}^{(-1)}\Phi\psi = 0$ . Subtracting this equation from  $\mathbf{G}\psi = 0$  we obtain that

$$(4.2.5) \quad (\mathbf{G} - \mathbf{G}^{(-1)}\Phi)\psi = 0.$$

Note that the last column of each matrix  $\Phi_j$  is  $(0, \dots, 0, 1)^{\text{tr}}$ , and hence the  $(1, d_1)$ -entry of  $\mathbf{G} - \mathbf{G}^{(-1)}\Phi$  is zero since the  $(1, d_1)$ -entry of the vector  $\mathbf{G}$  is 1. We further note that the  $(1, \sum_{i=1}^j d_i)$ -entry of  $\mathbf{G} - \mathbf{G}^{(-1)}\Phi$  is equal to

$$\mathbf{g}_{jd_j} - \mathbf{g}_{jd_j}^{(-1)} \text{ for } j = 2, \dots, m.$$

We claim that  $\mathbf{g}_{jd_j} - \mathbf{g}_{jd_j}^{(-1)} = 0$  for  $j = 2, \dots, m$ .

To prove the claim above, suppose on the contrary that there exists some  $2 \leq j \leq m$  for which  $\mathbf{g}_{jd_j} - \mathbf{g}_{jd_j}^{(-1)}$  is nonzero. We pick an  $N \in \mathbb{N}$  sufficiently large for which all entries of  $(\mathbf{G} - \mathbf{G}^{(-1)}\Phi)$  are regular at  $t = \theta^{q^N}$ , and  $\mathbf{g}_{jd_j} - \mathbf{g}_{jd_j}^{(-1)}$  is non-vanishing at  $t = \theta^{q^N}$ . Specializing (4.2.5) at  $t = \theta^{q^N}$  and using (4.2.2) we obtain a nontrivial  $\bar{k}$ -linear relations among  $Z_2^{q^N}, \dots, Z_m^{q^N}$  because the  $(1, d_1)$ -entry of  $\mathbf{G} - \mathbf{G}^{(-1)}\Phi$  is zero. By taking a  $q^N$ -th root we obtain a nontrivial  $\bar{k}$ -linear relation among  $Z_2, \dots, Z_m$ , whence a contradiction since we assume that  $Z_2, \dots, Z_m$  are linearly independent over  $\bar{k}$ .

Thus by (3.1.1) we have that  $\mathbf{g}_{jd_j} \in \mathbb{F}_q(t)$  for  $j = 2, \dots, m$ . Note that each entry of  $\mathbf{G}$  is regular at  $t = \theta$ . By specializing the equation  $\mathbf{G}\psi = 0$  at  $t = \theta$  and using (4.2.1) and (4.2.4), we obtain a nontrivial  $k$ -linear relation among  $Z_1, \dots, Z_m$ . This contradicts to our assumption, and hence we finish the proof.

## 5. LINEAR INDEPENDENCE OF MONOMIALS OF MULTIZETA VALUES

The main goal of this section is to give a proof of Theorem 2.2.1, which follows from Theorem 3.4.4. Thus, we shall show first that each nontrivial monomial of MZVs has the MZ property.

**5.1. Review of Anderson-Thakur theory.** We put  $D_0 := 1$ , and  $D_n := \prod_{i=0}^{n-1} (\theta^{q^n} - \theta^{q^i})$  for  $n \in \mathbb{N}$ . For any nonnegative integer  $n$ , we define the Carlitz factorial

$$\Gamma_{n+1} := \prod_i D_i^{n_i},$$

where

$$n = \sum n_i q^i \quad (0 \leq n_i \leq q-1)$$

is the base  $q$  expansion of  $n$ . The following theorem is the key ingredient to create suitable difference equations related to MZVs.

**Theorem 5.1.1.** (Anderson-Thakur, [AT90, 3.7.4] and [AT09, 2.4.1]) *For each nonnegative integer  $i$ , there exists a unique polynomial  $H_i(t) \in A[t]$  such that*

$$(H_{s-1} \Omega^s)^{(d)}(\theta) = \frac{\Gamma_s S_d(s)}{\tilde{\pi}^s}$$

for all nonnegative integers  $d$  and  $s \in \mathbb{N}$ .

**5.2. Difference equations associated to MZVs.** From now on, we fix an  $r$ -tuple  $(s_1, \dots, s_r) \in \mathbb{N}^r$ . In what follows, we review how Anderson and Thakur [AT09] were able to create a suitable difference equation  $\psi^{(-1)} = \Phi\psi$  associated to the multizeta value  $\zeta(s_1, \dots, s_r)$ .

For each  $1 \leq i \leq r$ , using Theorem 5.1.1 we define the polynomial

$$Q_{i+1,i} := H_{s_i-1} \in A[t],$$

and then form the matrix

$$Q := \begin{pmatrix} 1 & & & & \\ Q_{21} & 1 & & & \\ & \ddots & \ddots & & \\ & & Q_{r+1,r} & 1 & \end{pmatrix} \in \text{Mat}_{(r+1)}(\bar{k}[t]).$$

Define the diagonal matrix

$$D := \begin{pmatrix} (t-\theta)^{s_1+\dots+s_r} & & & & \\ & (t-\theta)^{s_2+\dots+s_r} & & & \\ & & \ddots & & \\ & & & (t-\theta)^{s_r} & \\ & & & & 1 \end{pmatrix} \in \text{Mat}_{(r+1)}(\bar{k}[t])$$

and put

$$\Phi := \Phi_{(s_1, \dots, s_r)} := Q^{(-1)} D \in \text{Mat}_{(r+1)}(\bar{k}[t]).$$

Define the series:

$$\begin{aligned}
 (5.2.1) \quad & L_2(t) := \sum_{i=0}^{\infty} (\Omega^{s_1} Q_{21})^{(i)} \\
 & L_3(t) := \sum_{i_1 > i_2 \geq 0} (\Omega^{s_2} Q_{32})^{(i_2)} (\Omega^{s_1} Q_{21})^{(i_1)} \\
 & \vdots \\
 & L_{r+1}(t) := \sum_{i_1 > \dots > i_r \geq 0} (\Omega^{s_r} Q_{r+1,r})^{(i_r)} \dots (\Omega^{s_1} Q_{21})^{(i_1)}
 \end{aligned}$$

By Theorem 5.1.1 one has that for each  $1 \leq j \leq r$ ,

$$L_{j+1}(\theta) = \frac{\Gamma_{s_1} \dots \Gamma_{s_j} \zeta(s_1, \dots, s_j)}{\tilde{\pi}^{s_1 + \dots + s_j}}.$$

Define the diagonal matrix

$$\Lambda := \begin{pmatrix} \Omega^{s_1 + \dots + s_r} & & & \\ & \Omega^{s_2 + \dots + s_r} & & \\ & & \ddots & \\ & & & \Omega^{s_r} \\ & & & & 1 \end{pmatrix} \in \text{Mat}_{(r+1)}(\mathcal{E}),$$

and put

$$L := \begin{pmatrix} 1 \\ L_2 \\ \vdots \\ L_{r+1} \end{pmatrix} \in \text{Mat}_{(r+1) \times 1}(\mathbb{C}_{\infty}[[t]]).$$

Finally, we define

$$(5.2.2) \quad \psi := \Lambda L \in \text{Mat}_{(r+1) \times 1}(\mathbb{C}_{\infty}[[t]])$$

and then we have the functional equation  $\psi^{(-1)} = \Phi \psi$  (cf. [AT09]). We first note that by [ABPo4, Prop. 3.1.3], each entry of  $\psi$  is in  $\mathcal{E}$ . We further note that

$$\psi(\theta) = \begin{pmatrix} \frac{1}{\tilde{\pi}^{s_1 + \dots + s_r}} \\ \vdots \\ \frac{\Gamma_{s_1} \dots \Gamma_{s_r} \zeta(s_1, \dots, s_r)}{\tilde{\pi}^{s_1 + \dots + s_r}} \end{pmatrix}.$$

**5.3. The key lemma.** The following lemma is to show that each MZV has the MZ property.

**Lemma 5.3.1.** *Given any  $r$ -tuple  $(s_1, \dots, s_r) \in \mathbb{N}^r$ , let  $L_2, \dots, L_{r+1}$  be the series defined in (5.2.1). Then for each  $1 \leq j \leq r$ , we have that*

$$L_{j+1}(\theta^{q^N}) = \left( \frac{\Gamma_{s_1} \dots \Gamma_{s_j} \zeta(s_1, \dots, s_j)}{\tilde{\pi}^{s_1 + \dots + s_j}} \right)^{q^N}$$

for any nonnegative integer  $N$ .

*Proof.* For the case  $N = 0$ , the result follows from the definition of  $L_{j+1}$  and Theorem 5.1.1. Now, let  $N$  be a positive integer. Fixing  $1 \leq j \leq r$ , we write  $L_{j+1} = L_{j+1}^{<N} + L_{j+1}^{\geq N}$ , where

$$L_{j+1}^{<N}(t) := \sum_{\substack{i_1 > \dots > i_j \geq 0; \\ i_j < N}} (\Omega^{s_j} Q_{j+1,j})^{(i_j)} \dots (\Omega^{s_1} Q_{21})^{(i_1)}$$

$$L_{j+1}^{\geq N}(t) := \sum_{i_1 > \dots > i_j \geq N} (\Omega^{s_j} Q_{j+1,j})^{(i_j)} \dots (\Omega^{s_1} Q_{21})^{(i_1)}.$$

Observe that using (3.1.2) one can express  $L_{j+1}^{<N}(t)$  as

$$L_{j+1}^{<N}(t) = \sum_{\substack{i_1 > \dots > i_j \geq 0; \\ i_j < N}} \frac{\Omega^{s_1 + \dots + s_j} Q_{j+1,j}^{(i_j)} \dots Q_{21}^{(i_1)}}{\left( (t - \theta^q) \dots (t - \theta^{q^{i_j}}) \right)^{s_j} \dots \left( (t - \theta^q) \dots (t - \theta^{q^{i_1}}) \right)^{s_1}}.$$

We claim that  $L_{j+1}^{<N}(\theta^{q^N}) = 0$ . To prove this claim, we first note that the order of vanishing of  $\Omega^{s_1 + \dots + s_j}$  at  $t = \theta^{q^N}$  is equal to  $s_1 + \dots + s_j$ . Moreover, each  $Q_{i+1,i}$  is a polynomial in  $A[t]$ , so is  $Q_{i+1,i}^{(n)}$  for every nonnegative integer  $n$ . On the other hand, we observe that each term in the expression of  $L_{j+1}^{<N}(t)$  above may have pole at  $t = \theta^{q^N}$  of order at worst  $s_1 + \dots + s_{j-1}$  since  $i_j < N$ . It follows that each term in the expression of  $L_{j+1}^{<N}(t)$  above has positive order of vanishing at  $t = \theta^{q^N}$ , whence the claim.

Therefore, we have that  $L_{j+1}(\theta^{q^N}) = L_{j+1}^{\geq N}(\theta^{q^N})$  (which we will see from the following that the series  $L_{j+1}^{\geq N}$  converges at  $t = \theta^{q^N}$ ). By definition, we can express  $L_{j+1}^{\geq N}$  as

$$L_{j+1}^{\geq N}(t) = \left( \sum_{i_1 > \dots > i_j \geq 0} (\Omega^{s_j} Q_{j+1,j})^{(i_j)} \dots (\Omega^{s_1} Q_{21})^{(i_1)} \right)^{(N)},$$

and hence

$$L_{j+1}^{\geq N}(\theta^{q^N}) = \left( \sum_{i_1 > \dots > i_j \geq 0} (\Omega^{s_j} Q_{j+1,j})^{(i_j)} \dots (\Omega^{s_1} Q_{21})^{(i_1)} \Big|_{t=\theta} \right)^{q^N} = \left( \frac{\Gamma_{s_1} \dots \Gamma_{s_j} \zeta(s_1, \dots, s_j)}{\tilde{\pi}^{s_1 + \dots + s_j}} \right)^{q^N},$$

where the second equality above comes from Theorem 5.1.1.  $\square$

**Corollary 5.3.2.** *Let  $\psi$  be defined as in (5.2.2). Then for any positive integer  $N$  we have*

$$\psi(\theta^{q^N}) = \begin{pmatrix} 0 \\ \vdots \\ \left( \frac{\Gamma_{s_1} \dots \Gamma_{s_r} \zeta(s_1, \dots, s_r)}{\tilde{\pi}^{s_1 + \dots + s_r}} \right)^{q^N} \end{pmatrix}.$$

*Proof.* It follows from the lemma above and the fact that  $\Omega(t)$  has simple zero at  $t = \theta^{q^N}$  for each  $N \in \mathbb{N}$ .  $\square$

**Corollary 5.3.3.** *Let  $Z$  be a multizeta value of weight  $w$ . If  $Z / \tilde{\pi}^w \in \bar{k}$ , then one has  $Z / \tilde{\pi}^w \in k$ .*

*Proof.* We write  $Z = \zeta(s_1, \dots, s_r)$  and let  $\Phi$  and  $\psi$  be constructed above (associated to  $\zeta(s_1, \dots, s_r)$ ). Define  $\tilde{\Phi} := [1] \oplus \Phi \in \text{Mat}_{r+2}(\bar{k}[t])$  and  $\tilde{\psi} := [1] \oplus \psi \in \text{Mat}_{(r+2) \times 1}(\mathcal{E})$ .

Then one has  $\tilde{\psi}^{(-1)} = \tilde{\Phi}\tilde{\psi}$  and the pair  $(\tilde{\Phi}, \tilde{\psi})$  satisfies the conditions of the ABP-criterion.

Since by hypothesis  $\zeta(s_1, \dots, s_r)/\tilde{\tau}^w := a \in \bar{k}^\times$ , by Theorem 3.2.1 there exists a vector  $P = (f, \dots, g) \in \text{Mat}_{1 \times (r+2)}(\bar{k}[t])$  so that

$$P\tilde{\psi} = 0, \text{ and } P(\theta) = (c, 0, \dots, 0, 1),$$

where  $c = -a\Gamma_{s_1} \cdots \Gamma_{s_r} \in \bar{k}^\times$ . Put  $\tilde{P} := \frac{1}{g}P = (f/g, \dots, 1)$ . Note that  $\tilde{P}\tilde{\psi} = 0$ . Using the  $(-1)$ -twisting operation on this equation and subtracting from the equation  $\tilde{P}\tilde{\psi} = 0$ , we have that

$$(5.3.4) \quad (\tilde{P} - \tilde{P}^{(-1)}\tilde{\Phi})\tilde{\psi} = 0.$$

Note that the last entry of the row vector  $\tilde{P} - \tilde{P}^{(-1)}\tilde{\Phi}$  is zero. Pick an integer  $N \gg 0$  so that all the entries of  $\tilde{P}$  are regular at  $t = \theta^{q^N}$ . By specializing (5.3.4) at  $t = \theta^{q^N}$  and using Corollary 5.3.2 we see that  $(f/g - (f/g)^{(-1)})(\theta^{q^N}) = 0$  for  $N \gg 0$ , whence  $f/g = (f/g)^{(-1)}$ . It follows from (3.1.1) that  $f/g \in \mathbb{F}_q(t)^\times$ . Thus, specializing the equation  $\tilde{P}\tilde{\psi} = 0$  at  $t = \theta$  shows the desired result.  $\square$

**5.4. Proof of Theorem 2.2.1.** Now, we give a proof of Theorem 2.2.1. For a positive integer  $r$ , we let  $\mathbb{N}^r$  be set of all  $r$ -tuples of positive integers. Let  $\mathfrak{A}$  be the free abelian group generated by the symbols as the  $r$ -tuples of positive integers for all  $r \in \mathbb{N}$ . More precisely, each nonzero element in  $\mathfrak{A}$  is of the form

$$b_1\mathfrak{s}_1 + \cdots + b_n\mathfrak{s}_n,$$

where  $\mathfrak{s}_i \in \mathbb{N}^{r_i}$  for some  $r_i \in \mathbb{N}$  ( $i = 1, \dots, n$ ) and for some not all zero integers  $b_1, \dots, b_n$ . We call a nonzero element in  $\mathfrak{A}$  *effective* if all the coefficients  $b_1, \dots, b_n$  are non-negative.

Now, we define a function  $\mathbf{Z} : \mathfrak{A} \setminus \{0\} \rightarrow k_\infty^\times$  as follows. First, given any  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$  we define

$$\mathbf{Z}(\mathfrak{s}) := \zeta(s_1, \dots, s_r).$$

Then for arbitrary  $\mathfrak{s} = \sum_{i=1}^n b_i\mathfrak{s}_i \in \mathfrak{A} \setminus \{0\}$  we define

$$\mathbf{Z}(\mathfrak{s}) := \prod_{i=1}^n \mathbf{Z}(\mathfrak{s}_i)^{b_i}.$$

Thus, any nontrivial monomial of MZVs can be expressed as  $\mathbf{Z}(\mathfrak{s})$  for some effective  $\mathfrak{s} \in \mathfrak{A} \setminus \{0\}$ .

Now, let the notation be as in Theorem 2.2.1. We write  $V_i = \{\mathbf{Z}(\mathfrak{a}_{i1}), \dots, \mathbf{Z}(\mathfrak{a}_{im_i})\}$  for some effective  $\mathfrak{a}_{ij} \in \mathfrak{A} \setminus \{0\}$  for  $j = 1, \dots, m_i$ ,  $i = 1, \dots, \ell$ . Given any MZV  $\zeta(s_1, \dots, s_r)$ , we have seen that  $\zeta(s_1, \dots, s_r)$  has the MZ property according to the construction in § 5.2 and Corollary 5.3.2. Therefore, each nontrivial monomial  $\mathbf{Z}(\mathfrak{a}_{ij})$  has the MZ property with weight  $w_i$  by Proposition 3.4.3 for  $j = 1, \dots, m_i$ ,  $i = 1, \dots, \ell$ . Therefore, the result of Theorem 2.2.1 follows by Theorem 3.4.4.

## 6. LINEAR INDEPENDENCE OF MONOMIALS OF CARLITZ MULTIPLE POLYLOGARITHMS

In [AT90], Anderson and Thakur showed that the Carlitz zeta value at  $n \in \mathbb{N}$  can be expressed as a  $k$ -linear combination of the  $n$ -th Carlitz polylogarithm at integral points in  $A$ . In this section, we first define the Carlitz multiple polylogarithms (abbreviated as CMPLs) and extend the work of Anderson-Thakur to multizeta values. We then show that the CMPLs at nontrivial algebraic points satisfy the MZ property and hence using Theorem 3.4.4 we derive a  $\bar{k}$ -linear independence for the different weight monomials of CMPLs at nontrivial algebraic points.

**6.1. Carlitz multiple polylogarithms.** We define  $\mathcal{L}_0 := 1$  and  $\mathcal{L}_i := \prod_{j=1}^i (\theta - \theta^{q^j})$  for  $i \in \mathbb{N}$ . For  $n \in \mathbb{N}$ , the  $n$ -th Carlitz polylogarithm is defined by

$$\log_n(z) := \sum_{i=0}^{\infty} \frac{z^{q^i}}{\mathcal{L}_i^n}.$$

(Note that in [AT90, Goss96] it is called the  $n$ -th Carlitz multilogarithm). It converges on the disc  $\left\{z \in \mathbb{C}_{\infty}; |z|_{\infty} < q^{\frac{nq}{q-1}}\right\}$ . Given any  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ , we define its associated Carlitz multiple polylogarithm as the following:

$$\text{Li}_{\mathfrak{s}}(z_1, \dots, z_r) := \sum_{i_1 > \dots > i_r \geq 0} \frac{z_1^{q^{i_1}} \cdots z_r^{q^{i_r}}}{\mathcal{L}_{i_1}^{s_1} \cdots \mathcal{L}_{i_r}^{s_r}}.$$

It converges on the polydisc

$$\left\{ (z_1, \dots, z_r) \in \mathbb{C}_{\infty}^r; |z_i|_{\infty} < q^{\frac{s_i q}{q-1}} \text{ for } i = 1, \dots, r \right\}.$$

**Proposition 6.1.1.** *Given any  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$  and any  $(u_1, \dots, u_r) \in (\mathbb{C}_{\infty}^{\times})^r$  with  $|u_i|_{\infty} < q^{\frac{s_i q}{q-1}}$  for  $i = 1, \dots, r$ , the value  $\text{Li}_{\mathfrak{s}}(u_1, \dots, u_r)$  is nonzero.*

*Proof.* Note that for any nonnegative integer  $i$  and positive integer  $n$ , we have

$$|\mathcal{L}_i^n|_{\infty} = q^{\frac{nq(q^i-1)}{q-1}}.$$

So the absolute value of the general term in the series  $\text{Li}_{\mathfrak{s}}(u_1, \dots, u_r)$  is given by

$$q^{\frac{q}{q-1}(s_1 + \dots + s_r)} |u_1 / (\theta^{\frac{q s_1}{q-1}})|_{\infty}^{q^{i_1}} \cdots |u_r / (\theta^{\frac{q s_r}{q-1}})|_{\infty}^{q^{i_r}}.$$

The hypotheses on  $(u_1, \dots, u_r)$  imply that the general term has a unique maximal absolute value when  $(i_1, \dots, i_r) = (r-1, \dots, 0)$ . The desired result thus follows from non-archimedean analysis.  $\square$

**6.2. Stuffle relations.** There are natural algebraic relations among the CMPLs at *nontrivial algebraic points*. For example, the classical stuffle relations for multiple polylogarithms work here.

Given  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$  and  $\mathfrak{s}' = (s'_1, \dots, s'_{r'}) \in \mathbb{N}^{r'}$ , fix a positive integer  $r''$  with  $\max\{r, r'\} \leq r'' \leq r + r'$ . We consider a pair consisting of two vectors  $\mathbf{v}, \mathbf{v}' \in \mathbb{Z}_{\geq 0}^{r''}$  which are required to satisfy  $\mathbf{v} + \mathbf{v}' \in \mathbb{N}^{r''}$  and which are obtained from the following ways. One vector  $\mathbf{v}$  is obtained from  $\mathfrak{s}$  by inserting  $(r'' - r)$  0s in all possible ways (including

in front and at end), and another vector  $\mathbf{v}'$  is obtained from  $\mathbf{s}'$  by inserting  $(r'' - r')$  0s in all possible ways (including in front and at end).

One observes from the definition of the series that the CMPLs satisfy the stuffle relations which are analogous to the classical case (cf. [Wo2]):

$$(6.2.1) \quad \text{Li}_{\mathbf{s}}(\mathbf{z})\text{Li}_{\mathbf{s}'}(\mathbf{z}') = \sum_{(\mathbf{v}, \mathbf{v}')} \text{Li}_{\mathbf{v}+\mathbf{v}'}(\mathbf{z}''),$$

where the pair  $(\mathbf{v}, \mathbf{v}')$  runs over all the possible expressions as above for all  $r''$  with  $\max\{r, r'\} \leq r'' \leq r + r'$ . For each such  $\mathbf{v} + \mathbf{v}' \in \mathbb{N}^{r''}$ , the component  $z''_i$  of  $\mathbf{z}''$  is  $z_j$  if the  $i$ th component of  $\mathbf{v}$  is  $s_j$  and the  $i$ th component of  $\mathbf{v}'$  is 0, it is  $z'_\ell$  if the  $i$ th component of  $\mathbf{v}$  is 0 and the  $i$ th component of  $\mathbf{v}'$  is  $s'_\ell$ , and finally it is  $z_j z'_\ell$  if the  $i$ th component of  $\mathbf{v}$  is  $s_j$  and the  $i$ th component of  $\mathbf{v}'$  is  $s'_\ell$ .

For example, for  $r = r' = 1$  (6.2.1) yields

$$\text{Li}_{\mathbf{s}}(\mathbf{z})\text{Li}_{\mathbf{s}'}(\mathbf{z}') = \text{Li}_{(s, s')}(z, z') + \text{Li}_{(s', s)}(z', z) + \text{Li}_{s+s'}(zz').$$

For  $r = 1, r' = 2$ , one has

$$\begin{aligned} \text{Li}_{\mathbf{s}}(\mathbf{z})\text{Li}_{(s'_1, s'_2)}(z'_1, z'_2) &= \text{Li}_{(s, s'_1, s'_2)}(z, z'_1, z'_2) + \text{Li}_{(s'_1, s, s'_2)}(z'_1, z, z'_2) + \text{Li}_{(s'_1, s'_2, s)}(z'_1, z'_2, z) \\ &\quad + \text{Li}_{(s+s'_1, s'_2)}(zz'_1, z'_2) + \text{Li}_{(s'_1, s+s'_2)}(z'_1, zz'_2). \end{aligned}$$

**6.3. The formula.** Since we are interested in the nonzero values of  $\text{Li}_{\mathbf{s}}$  at algebraic points, according to the non-vanishing result above we simply call any such  $(u_1, \dots, u_r) \in (\bar{k}^\times)^r$  with  $|u_i|_\infty < q^{\frac{s_i q}{q-1}}$  for  $i = 1, \dots, r$ , a *nontrivial algebraic point*. The following result is a generalization of Anderson-Thakur [AT90] to MZVs.

**Theorem 6.3.1.** *For any  $r$ -tuple  $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ , the multizeta value  $\zeta(s_1, \dots, s_r)$  is a  $k$ -linear combination of  $\text{Li}_{\mathbf{s}}$  at some points in  $A^r$ .*

*Proof.* We recall the series  $L_{r+1}(t)$  defined in (5.2.1). Using (3.1.2) we can rewrite it as

$$L_{r+1}(t) = \Omega^{s_1 + \dots + s_r}(t) \sum_{i_1 > \dots > i_r \geq 0} \frac{H_{s_{r-1}}^{(i_r)}(t) \cdots H_{s_1-1}^{(i_1)}(t)}{\left((t - \theta^q) \cdots (t - \theta^{q^{i_r}})\right)^{s_r} \cdots \left((t - \theta^q) \cdots (t - \theta^{q^{i_1}})\right)^{s_1}},$$

where  $H_{s_i-1} \in A[t]$  for  $i = 1, \dots, r$ . Recall that

$$L_{r+1}(\theta) = \frac{\Gamma_{s_1} \cdots \Gamma_{s_r} \zeta(s_1, \dots, s_r)}{\tilde{\pi}^{s_1 + \dots + s_r}}$$

Since  $\Omega(\theta) = 1/\tilde{\pi}$ , we see that

$$(6.3.2) \quad \Gamma_{s_1} \cdots \Gamma_{s_r} \zeta(s_1, \dots, s_r) = \frac{L_{r+1}(\theta)}{\Omega(\theta)^{s_1 + \dots + s_r}} = \sum_{i_1 > \dots > i_r \geq 0} \frac{H_{s_{r-1}}^{(i_r)}(\theta) \cdots H_{s_1-1}^{(i_1)}(\theta)}{\mathcal{L}_{i_r}^{s_r} \cdots \mathcal{L}_{i_1}^{s_1}}.$$

Pick an integer  $\ell \gg 0$  for which each  $H_{s_j-1}$  can be written as

$$H_{s_j-1}(t) = a_{j\ell} t^\ell + \cdots + a_{j1} t + a_{j0} \in A[t].$$

Thus we have

$$H_{s_j-1}^{(i_j)}(\theta) = a_{j\ell}^{q^{i_j}} \theta^\ell + \cdots + a_{j1}^{q^{i_j}} \theta + a_{j0}^{q^{i_j}}.$$



It follows that  $H_{s_{r-1}}^{(i_r)}(\theta) \cdots H_{s_1-1}^{(i_1)}(\theta)$  can be expressed as an  $A$ -linear combination of

$$a_{rh_r}^{q^{i_r}} \cdots a_{1h_1}^{q^{i_1}}$$

with  $1 \leq h_j \leq \ell$  for  $j = 1, \dots, r$ . Therefore, dividing both sides of (6.3.2) by  $\Gamma_{s_1} \cdots \Gamma_{s_r}$  implies the desired result.  $\square$

**6.4. Linear independence.** The following lemma establishes that the Carlitz multiple polylogarithms at *nontrivial algebraic points* satisfy the MZ property.

**Lemma 6.4.1.** *Given any  $r$ -tuple  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ , we let  $(u_1, \dots, u_r) \in (\bar{k}^\times)^r$  satisfy  $|u_i|_\infty < q^{\frac{s_i q}{q-1}}$  for  $i = 1, \dots, r$ . Then  $\text{Li}_{\mathfrak{s}}(u_1, \dots, u_r)$  has the MZ property with weight  $\sum_{i=1}^r s_i$ .*

*Proof.* For each  $1 \leq j \leq r$ , we define

$$\mathfrak{L}_{j+1}(t) := \sum_{i_1 > \cdots > i_j \geq 0} (\Omega^{s_j} u_j)^{(i_j)} \cdots (\Omega^{s_1} u_1)^{(i_1)},$$

which can be also written as

$$\mathfrak{L}_{j+1}(t) = \Omega^{s_1 + \cdots + s_j} \sum_{i_1 > \cdots > i_j \geq 0} \frac{u_j^{q^{i_j}} \cdots u_1^{q^{i_1}}}{\left( (t - \theta q) \cdots (t - \theta q^{i_j}) \right)^{s_j} \cdots \left( (t - \theta q) \cdots (t - \theta q^{i_1}) \right)^{s_1}}.$$

One sees that  $\mathfrak{L}_{j+1}$  converges on  $|t|_\infty < q^q$ , and  $\mathfrak{L}_{r+1}(\theta) = \text{Li}_{\mathfrak{s}}(u_1, \dots, u_r) / \tilde{\pi}^{s_1 + \cdots + s_r}$ .

Define

$$\Omega := \begin{pmatrix} 1 & & & \\ u_1 & 1 & & \\ & \ddots & \ddots & \\ & & u_r & 1 \end{pmatrix} \in \text{Mat}_{(r+1)}(\bar{k}[t])$$

and

$$\mathfrak{L} := (1, \mathfrak{L}_2, \dots, \mathfrak{L}_{r+1})^{\text{tr}} \in \text{Mat}_{(r+1) \times 1}(\mathbb{C}_\infty[[t]]).$$

Let  $D \in \text{Mat}_{(r+1)}(\bar{k}[t])$  and  $\Lambda \in \text{Mat}_{(r+1)}(\mathcal{E})$  be the diagonal matrices constructed in § 5.2. Put

$$\Phi := \Phi_{\mathfrak{s}} := \Omega^{(-1)} D, \quad \psi := \psi_{\mathfrak{s}} := \Lambda \mathfrak{L}.$$

Then one has  $\psi^{(-1)} = \Phi \psi$  and by [ABPo4, Prop. 3.1.3] all the entries of  $\psi$  are in  $\mathcal{E}$ . One immediately observes that  $(\Phi, \psi, \text{Li}_{\mathfrak{s}}(u_1, \dots, u_r))$  satisfies (1) – (3) of the Definition 3.4.1.

Using the same argument as in Lemma 5.3.1 one sees that for any positive integer  $N$ ,

$$\mathfrak{L}_{j+1}(\theta^{q^N}) = \left( \frac{\text{Li}_{(s_1, \dots, s_j)}(u_1, \dots, u_j)}{\tilde{\pi}^{s_1 + \cdots + s_j}} \right)^{q^N}.$$

for every  $1 \leq j \leq r$ . Again using the fact that  $\Omega(t)$  has simple zero at  $t = \theta^{q^N}$  for every  $N \in \mathbb{N}$ ,  $(\Phi, \psi, \text{Li}_{\mathfrak{s}}(u_1, \dots, u_r))$  satisfies (4) of the Definition 3.4.1, whence the result.  $\square$

**Definition 6.4.2.** *Given any  $r$ -tuple  $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ , let  $Z$  be a specialization of  $\text{Li}_{\mathfrak{s}}$  at some nontrivial algebraic point. We define the weight of  $Z$  to be  $\text{wt}(Z) := s_1 + \cdots + s_r$ .*

Let  $Z_1, \dots, Z_n$  be specializations of some Carlitz multiple polylogarithms at nontrivial algebraic points. We define the weight of the monomial  $Z_1^{m_1} \dots Z_n^{m_n}$  to be

$$\sum_{i=1}^n m_i \text{wt}(Z_i).$$

Let  $\overline{\mathfrak{M}}_w$  be the  $\bar{k}$ -vector space spanned by all weight  $w$  CMPLs at nontrivial algebraic points, and let  $\overline{\mathfrak{M}}$  be the  $\bar{k}$ -algebra generated by all CMPLs at nontrivial algebraic points. The main result in this section is stated as follows.

**Theorem 6.4.3.** *Let  $w_1, \dots, w_\ell$  be  $\ell$  distinct positive integers. Let  $V_i$  be a finite set consisting of weight  $w_i$  monomials of some specializations of Carlitz multiple polylogarithms at nontrivial algebraic points for  $i = 1, \dots, \ell$ . If  $V_i$  is a linearly independent set over  $k$ , then*

$$\{1\} \bigcup_{i=1}^{\ell} V_i$$

*is linearly independent over  $\bar{k}$ . In particular,*

$$\overline{\mathfrak{M}} = \bar{k} \oplus_{w \in \mathbb{N}} \overline{\mathfrak{M}}_w \text{ and } \overline{\mathfrak{M}} \text{ is defined over } k.$$

*Proof.* By Lemma 6.4.1 each specialization of Carlitz multiple polylogarithm at a nontrivial algebraic point has the MZ property. It follows that by Proposition 3.4.3 each nontrivial monomial of such values has the MZ property. Therefore, the result follows from Theorem 3.4.4.  $\square$

By (6.2.1) and Theorem 6.4.3, one sees that the  $\bar{k}$ -algebra  $\overline{\mathfrak{M}}$  is a graded algebra (graded by weights) defined over  $k$ .

**Corollary 6.4.4.** *Every nontrivial monomial of Carlitz multiple polylogarithms at nontrivial algebraic points is transcendental over  $k$ .*

**Corollary 6.4.5.** *The ratio of two different weight nontrivial monomials of Carlitz multiple polylogarithms at nontrivial algebraic points is transcendental over  $k$ .*

**Corollary 6.4.6.** *Let  $Z_1, Z_2$  be two nonzero values which are Carlitz multiple polylogarithms at algebraic points. Suppose that  $Z_1$  and  $Z_2$  are of the same weight. Then either the ratio  $Z_1/Z_2$  is in  $k$  or  $Z_1$  and  $Z_2$  are algebraically independent over  $k$ .*

*Proof.* The proof is essentially the same as the proof of Theorem 2.3.2.  $\square$

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